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Housing Equity Withdrawal in the Portfolio Choice for Deferred Pension Income

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Abstract: The paper examines the optimal structure of retirement incomes from the perspective of the young pessimistic decision maker, i.e. a young employee who can invest in different pension funds or in residential property and who is exposed to volatility of interest rate. After a worldwide shift from defined benefit (DB) systems to defined contribution (DC) schemes, these risks are squarely on the workers' shoulders, as also explained in the recently published CEJOR paper of Di Giacinto and Vigna. Therefore, the risks should be mitigated by a proper portfolio structure. Because occupational and private pension incomes decrease with the decreasing interest rate, the only available investment strategy to mitigate these risks is to invest also in the assets whose cash flow is negatively correlated with the interest rate. Therefore the advantage to invest in one's own home to be able to participate later, i.e. after retirement, in Equity Release Schemes (ERS) is highlighted and the optimal structure of pension incomes is derived.

We present how the variance of the expected pension income due to the variance of the interest rate at retirement of the decision maker could be reduced when using residential property as the 4th pillar of the pension system. From the pessimistic perception the optimal structure of a portfolio of pension wealth is derived, when the criterion function is a combination of the minimum variance of expected income and the minimum difference between the critical value, determined by the pessimistic decision maker, and the amount needed for financing one's desired lifestyle in retirement. We also present some remarks available as the consequences of this optimization procedure, like (a) the more reliable the wherewithal for financing one's targeted lifestyle, the higher the share of assets in funded private pension schemes and the lower in ERS; (b) the more risk-averse a decision maker, the less stable the optimal structure of pension wealth; and (c) the optimal structure of pension wealth portfolio of a pessimistic decision maker remains constant and very close to the optimal structure at the minimum variance of the total portfolio, and does not depend much on the risk aversion of the decision maker when the certain ratio between target retirement income and the accumulated pension wealth is achieved.

Keywords *decision making. pessimistic decision . Equity Release Scheme (ERS) . longevity risk . reverse mortgage . portfolio optimization . pension.*

1 Introduction to the pessimistic approach in the creation of the pension portfolio

According to the CEJOR article of Prettner and Prskawetz (2010), population ageing can have positive impacts on economic growth if it triggers additional savings while leaving the size of the workforce unchanged; it can have negative impacts on economy and welfare if the size of the workforce decreases relative to the amount of retirees. But by reducing the welfare of older cohorts their purchasing power decreases also. This phenomenon additionally reduces the employment opportunities of young Europeans. Apparently, the decreasing pension incomes in many EU Pay As You Go (PAYG) public pension systems and the transition from the defined benefit (DB) to the defined contribution (DC) funded pension schemes affect the level of anxiety in prospective retirees and, as a result, pessimistic projections and the pessimistic approach to the creation of the pension portfolio.

As mentioned in the CEJOR paper of Di Giacinto and Vigna (2012), the crisis of national PAYG public pension systems across Europe caused by the ageing of population is forcing governments of most European countries to drastically cut pension benefits of future generations while it encourages the development of other three pillars of the pension system, i.e. fully funded occupational pension schemes, private pension schemes and, in some European countries, also Equity Release Schemes (ERS) as housing equity withdrawals from the residential property owned by the elderly. In a few decades the future retirees will have to rely heavily on the retirement incomes of those pillars if they want to maintain the pre-retirement standard of living. Pension schemes can be either defined benefit (DB) or defined contribution (DC). While in DB pension schemes, benefits are based on a simple formula given in the rules and do not depend on the investment experience of the scheme, DC pension scheme leave all risk of turbulent economic environment on the shoulders of workers who contribute to the scheme and who are the final beneficiaries. For example, such changes brought to new US retirees, i.e. who retired in the last decade, less than 75% the amount of income compared to older retirees, whose schemes were contracted as DB schemes, though they contributed equal premiums.

In funded defined contribution pension systems, the amount of the yearly pension depends on the accumulated amount in the individual retirement account and long-term interest rate at the moment of retirement. As presented by Shiller[1] (see Fig. 1), the long-term interest rates are highly volatile in the USA, while in Europe, countries have often ended in hyperinflation over the history, and hence the secular trend is difficult to analyse. We can see

[1] http://www.irrationalexuberance.com, 2005; updated data December 2013

how the yearly amount of one's pension is uncertain. Markowitz (1952) has shown that such volatility of returns can be mitigated by adding negatively correlated assets in the portfolio.

Our paper examines the possibility of developing and implementing flexible Housing Equity Release Schemes (ERS), named also Housing Equity Withdrawal, as a means of providing a more stable welfare provision for the elderly, adding housing wealth in a portfolio of pension instruments from the perception of the pessimistic decision maker who wishes to avoid poverty, determined in advance by a certain lifestyle line or, in general, by the amount needed for financing his desired lifestyle in retirement. We had assume that the portfolio of more diversified sources of income after retirement could provide safer retirement provision for older persons by stabilizing the total disbursement to the beneficiary, where the volatility depends on the volatility of the interest rate (funded pensions). Namely, the falling interest rates since 2009 have had adverse effects on funded pensions, and the austerity policies since 2010 have reinforced the inequalities, particularly among the 'income poor' but 'asset rich' older inhabitants. In many European countries pensions are often not sufficient to cover health care expenses and other basic needs of older persons, who often also have difficulties paying property taxes.

Hence, it is crucial to solve the question of an appropriate welfare mix for older people. Therefore, the authors seek to examine the following research questions:

(a) How to ensure a more appropriate welfare mix for older people to avoid the fall below the desired standard of living;

(b) What role could Equity Release Schemes play for the pessimistic decision maker; and

(c) How should the financial industry develop attractive financial products to fit within the mix of other (private and public) retirement provisions, where the risks of falling below the targeted life style line due to the falling interest rate can be mitigated to the benefit of older (pessimistic) property owners?

Using actuarial mathematics with life contingencies, the paper will present how reverse mortgage systems (ERS loan model) with the embedded insurance for longevity might improve the satisfaction of inhabitants. Based on the presented findings and on the Portfolio Theory (Markowitz, 1952; Tu and Zhou, 2011), we also show that the interest rate variation, which can reduce the income of older persons under the targeted lifestyle line, has a significantly smaller impact on the welfare of the elderly if the first three pension pillars are combined with the ERS loan model. Such financial product would be more attractive in the insurance and banking industries.

Insurance issues are viewed within the wider context of risk management. The early discussions of operational research (OR) and its impact on major areas of insurance are available by Haehling von Lanzenauer and Wright (1991) with an extended list of references to actuarial science in early papers on insurance studies in the OR literature. Three components of asset/liability management are described in their paper: (1) a multi-stage stochastic program for coordinating the asset/liability decisions; (2) a scenario generation procedure for modelling the stochastic parameters; and (3) solution algorithms for solving the resulting large-scale optimization problem. In the last decade, there increased the number of papers on OR models in life insurance, especially from the perspective of Life and Health insurance industry (Wu et al., 2007); in many articles bi-objective dynamic programming or simulation models have been developed also for insurance industry (Josa-Fombellida and Rincón-Zapatero, 2008; Ermolieva, 2005), but there is a shortage of optimisation models where the optimal decisions from the perception of insured persons would be available.

Markowitz discussed MV approximations to expected utility (Markowitz, 2014). He critically evaluated the pros and cons on the assumptions of the contemporary portfolio theory, and argued some attacks related to the confusion of sufficient versus necessary conditions for applying the theory. He has given an overview of a half a century of research on MV approximations to expected utility. However, in the special issue dedicated to the 60th anniversary of Portfolio Theory only one article discussed the optimization problems in management and control of pension systems. Bae, Kim and Mulvey (the same issue, 2014) tried to obtain multiple distributions that could convert the static MV model into an optimization problem under uncertainty. The authors used a stochastic programming approach to optimize portfolios under a regime switching framework. They explored the scenario generation to mathematically formulate the optimization problem that is applied to a pension fund in order to investigate the behaviour of the optimal solution over time. The authors concluded that the regime information helps portfolios avoid risk during left-tail events.

In our paper, we shall further study the problem of left-tail events in the pension portfolio, i.e. by changing the regimes you can avoid risk during accumulation and de-accumulation period, having in mind the real property in the portfolio of the decision maker. Therefore the paper first describes the existing models of ERS, and then it introduces the model for mitigating the credit default risk using the longevity insurance.

Based on the data of volatility of interest rate in USA (Shiller, 2005+) we present how the variance of the expected pension income due to the variance of the interest rate at retirement of the decision maker could be reduced when using residential property as the 4th pillar of the pension system. From the pessimistic perception the optimal structure of a portfolio of pension wealth is derived, where the criterion function is a combination of the minimum variance of expected income and the minimum difference between the critical value, determined by the pessimistic decision maker, and the amount needed for financing one's desired lifestyle in retirement. We also present some remarks available as the consequences of these optimization procedures like:

(a) the more reliable the wherewithal for financing one's targeted lifestyle, the higher the share of assets in funded private pension schemes and the lower in ERS;

(b) the more risk-averse a decision maker, the less stable the optimal structure of pension wealth; and

(c) the optimal structure of pension wealth portfolio of a pessimistic decision maker remains constant and very close to the optimal structure at the minimum variance of the total portfolio, and does not depend much on the risk aversion of the decision maker when the certain ratio between target retirement income and the accumulated pension wealth is achieved.

In this article other uncertainties like changing locality and demographic factors which influence value of properties have not been considered. The paper intends to show that regarding his old-age welfare protection, a young person, when first employed, should consider buying a home instead of renting one, in order to accumulate real estate assets. The numerical examples are also presented.

2 Modelling ERS

ERSs transform fixed assets in owner occupied dwellings into liquid assets for private pensions. They thus enable a homeowner to access the wealth accumulated in the form of the home, while being able to continue to live in it. An illiquid asset becomes a source of liquidity, mainly for consumption needs. ERS can take two different forms:

(a) Loan Model ERS, also known as reverse mortgage provide a loan that will be repaid from the sale of property mainly after the death of owner; and

(b) Sale Model ERS, which involve an immediate sale of the property but provide for the right to remain in occupation and to use the cash price for income in retirement.

ERS must therefore:

(a) be a financial service;

(b) be a source of liquidity for the future;

(c) contain a strong entitlement to remain in occupation of the property; and

(d) rely solely on the sale of the property for repayment/payment of the funds released to be used as a retirement pension.

Payments take the form of a lump sum or periodic (monthly, yearly) income, and are either secured by means of a mortgage on the property or generated by an immediate sale. Under the Loan Model, repayment is made from the proceeds of the sale of the property either after death of the homeowner or when the property has become vacated for a longer time (see details on European implementations in Reifner et al., 2009).

Sale model: The sale model is very straightforward and is similar to the general annuity model (funded pensions, i.e. 2nd and 3rd pillars), as described in Gerber (1980). In the sale model, the whole value of the real estate is transferred to lifetime annuity at the moment of closing of the ERS contract. The value of the property is used for purchase of the lifetime annuity. Let us replace the standard notation used in actuarial mathematics as follows:

$a(x) = \ddot{a}_x$	actuarial notation for the present value of the lifetime annuity in amount of 1 EUR, paid at the beginning of each year for the person that is x years old - according to the mortality table;
$p(j\|x) = {}_j p_x$	the probability that the person that is x years old will survive the next j years;
$\nu = 1/(1+i)$	discounting factor where i is the annual interest rate

The amount of yearly disbursement of the lifetime annuity therefore needs to cover the interest on the principal amount taken out and the yearly annuity paid to the beneficiary of the ERS, as is presented in equation (1):

$$a(x) = \sum_{j=0}^{110-x} p(j|x) \cdot v^j = \sum_{j=0}^{110-x} p(j|x) \cdot (1/(1+i))^j \tag{1}$$

The amount of lifetime annuity is calculated as the annuity factor multiplied by the net value of real estate, which is calculated as the value of the real estate minus the cost associated with the transaction (valuation costs, taxes, costs of sale).

We shall further use more simple expression as annuity factor $fr(x,i)$, which is:

$$fr(x,i) = 1/\{(1+\gamma_2)\cdot a(x)\} = 1/\left\{(1+\gamma_2)\sum_{j=0}^{110-x} p(j|x)\cdot(1/(1+i))^j\right\} \qquad (2)$$

where the rate γ_2 represents the costs associated with the disbursement of the annuity that the insurance company charges for each pay-out in the period of annuity.

The yearly amount of annuity R_h is calculated according to the value of real estate *VRE* and annuity factor $fr(x,i)$:

$$R_h = fr(x,i)\cdot(VRE - C) \qquad (3)$$

Here C are all costs associated with closing the ERS contract and with the sale of the property after the death of the beneficiary.

Loan model: The loan model or 'reverse mortgage' is a type of home loan that allows a borrower to open up a line of credit using his home as collateral. With the loan model the beneficiary draws liquid amounts in lump-sum or/and periodically from the value of the real estate in the form of a loan secured by a mortgage on the real estate. With the part of this liquid amount that is drawn from the real estate the beneficiary purchases deferred lifetime annuity in the form of a monthly (or yearly) premium. In this way the beneficiary insures his longevity so that if he lives longer than his life expectancy he will receive a lifetime annuity until his death. In the paper, we propose the ERS model with the insurance for longevity, where the periodic disbursement that the beneficiary receives is the difference between the amount drawn and the annuity premium for longevity insurance. In this way, if the beneficiary survives the drawing period of ERS *(n* years), he receives a lifetime annuity that covers the disbursement to the beneficiary and the interest on the outstanding loan. This is a new scheme, first proposed in Bogataj (2013). Generally, loan models allow the beneficiary to draw the value of the real estate in different ways:

(a) In lump sum at the closing of the ERS contract;

(b) In the form of line of credit so that he can draw it when necessary;

(c) In uniform periodic amounts in the period of life expectancy.

The maximum amount of loan (*MLA*) that can be drawn from the real estate is the value of the real estate *(VRE)* minus all the costs *(C),* i.e. those associated with closing the ERS contract (C_1) and with the sale of the property after the death of the beneficiary (C_2).

$$MLA = VRE - C = VRE - C_1 - C_2 \quad (4)$$

A life annuity consists of a series of payments which are made while the beneficiary (of initial age *x*) lives.

The present value of the life annuity due with yearly payments at the beginning of each year in the amount of 1 EUR in next n years for person which is x years old, is denoted by $a(x|n) = \ddot{a}_{x:\overline{n}|}$, where the following equation can be written:

$$a(x|n) = \sum_{j=0}^{n} p(j|x)v^j \quad (5)$$

The present value of the life annuity deferred for *n* years with yearly payments in the amount of 1 EUR is denoted by $b(x,\text{n}) = {}_{n|}\ddot{a}_x$, where the following equation can be written:

$$b(x,n) = p(n|x)\cdot v^n \cdot a(x) = p(n|x)\cdot v^n \cdot \sum_{j=0}^{110-(x+n)} p(j|x+n)\cdot v^j \quad (6)$$

The premium rate for longevity insurance $prs(x,i,n)$ is:

$$prs(x,i,n) = (1+\gamma_2)\cdot b(x,n) / (1+\gamma_1)\cdot a(x|n) \quad (7)$$

where γ_1 represents the rate of administration expenses that are charged against the policy in the period of premium payments and γ_2 represents the rate of administration expenses that are charged against the policy in the period of annuity payments. The yearly amount of premium *(PR)* is calculated as:

$$PR = prs(x,i,n)\cdot R_h\,, \quad (8)$$

where R_h is the annuity amount withdrawn from the housing asset. In this case, the yearly disbursement amount *(YDA*) that the beneficiary can draw from the real estate is:

$$YDA = i\cdot MLA / \left[(1+i)^n - 1\right] = YPA + PR = YPA + prs(x,i,n)\cdot(YPA + MLA\cdot i) \quad (9)$$

3 Mitigating the credit default risk and avoiding the fall of the pension income below the lifestyle line

3.1 Credit default risk in case of ERS

The main risks concerning ERS that can cause credit default are:

- the uncertain longevity of the owner occupier, realized when the value of a property being sold does not cover the amount of loan,
- the risk of increase in interest rates, and
- depreciation in the value of the property,

though some political changes could also influence the value of property (Černe et al., 2012).

Deferred annuity as an insurance for longevity is already used in insurance industry, but not in combination with reverse mortgage as the one proposed here. Sustainability and market consistency regarding longevity are among the main concerns of The Actuarial Association of Europe [2] .

Without an effective insurance for longevity, real estate cannot be used as the 4th pension pillar, because the equity release without longevity insurance presents a great risk for the provider of reverse mortgage (bank) and also for the beneficiary. For the reverse mortgage provider, there is the risk that the value of the loan together with accrued interest will be greater than the value of real estate in case of the death of the beneficiary. For the beneficiary, there is the risk that he will live longer than the agreed period of drawing liquid amounts that is defined in the reverse mortgage loan contract. To avoid exposure to these risks, a safe reverse mortgage contract also needs to include a kind of insurance for longevity. This insurance can be provided in three ways:

(a) Through public finance so that the risk is socialized and the management of risk is assumed by the government (as is the case in the USA where The Home Equity Conversion Mortgages Insurance is a clear example of such a scheme. Lenders under this program are protected against losses arising when the loan balance exceeds the value of real estate at time of settlement. But because of implicit government guarantees underlying this insurance, it may become a serious drain on the fiscus, as the market expands after crisis, as described also by Wang et al. (2007) and Chen et al. (2010);

(b) The risk can be transferred to insurance companies.

[2] See the articles at: www. actuary.eu, where the first author is member of Pension Committee.

According to the results in Blake et al. (2013), the huge economic significance of the longevity risk has begun to be recognized and quantified in their article, presenting the birth and development of the Life Market, the new market related to the transfer of longevity and mortality risks. The authors note that the emergence of a traded market in longevity-linked capital market instruments could act as a catalyst to help facilitate the development of annuity markets, both in the developed and the developing worlds, and protect the long-term viability of retirement income provision globally. The possible instruments have been studied and developed also in Lee et al. (2012) and Yang and Wang (2013);

(c) The third way is possible through a mutual insurance company, as proposed in Bogataj (2013) and inherently present in this paper. The risk of longevity can be mitigated by the use of annuity insurance where legal, obligatory mortality tables in Slovenia are German *DAV1994R*, which are close to mortality projection 2050; hence, insurance companies benefit on overestimation of longevity today. Indeed, it is better to share the benefit of this overestimation among the remaining seniors, but it is difficult to avoid the impact of the volatile interest rate.

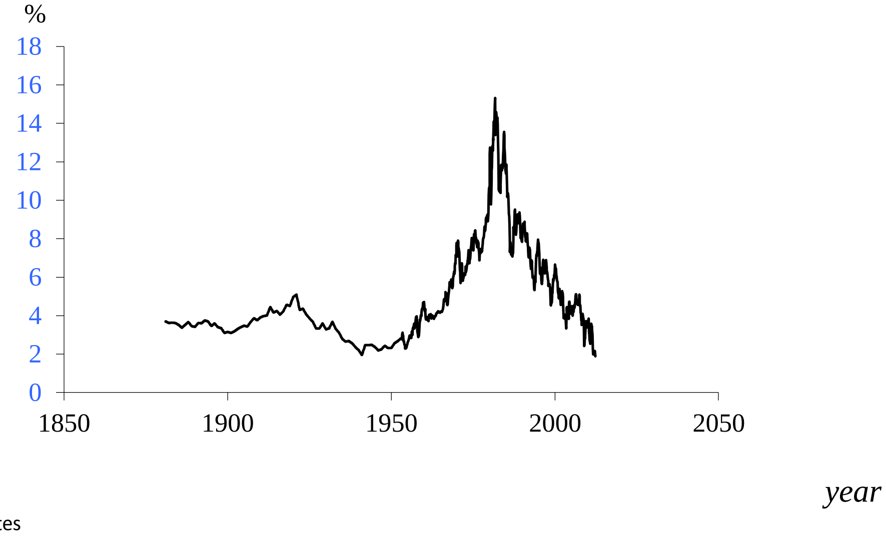

Figure 1: Interest rate (in % : 1871–2012) in USA

The volatility of the interest rate from 1871 to 2012 was very high even in the USA. It is presented in Figure 1.

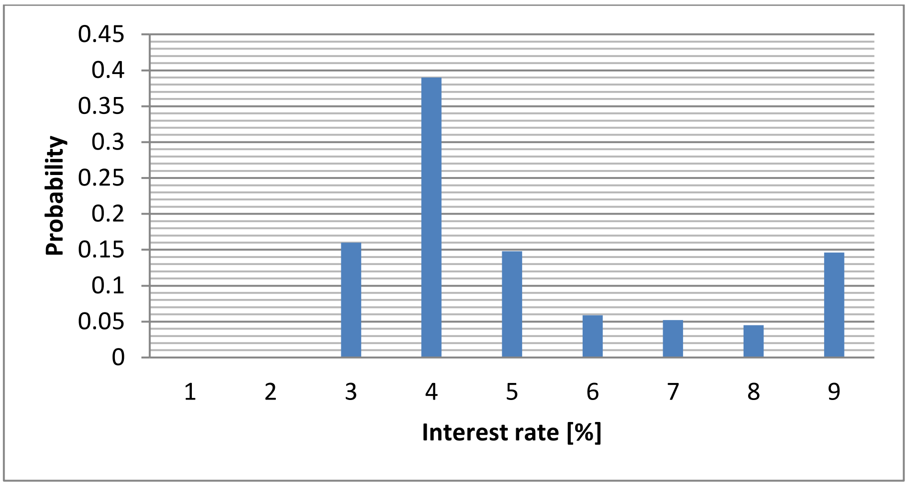

Figure 2*:* Distribution of 22-year (264 months) moving averages for the yearly interest rate from 1871 to 2012 (22 years is an approximate life expectancy for a 65-year old man, after 2050 according to EUROSTAT population projections in EU.

From Shiller's time series of the interest rate in USA, presented in Figure 1, we calculated the 22-year (264 months) moving average of the interest rate (see Figure 2). The most frequent moving average of the interest rate was equal to 4, while the average and variance were more than 5.

3.2 ERS in the portfolio of assets after retirement

The volatility of interest rate does not influence the PAYG systems, therefore the yearly pension income from this system can be denoted by constant *A*. From (3) it follows that annuity R_p of private pension increases with increasing i. Also, occupational funded pensions R_o depend on the interest rate in the same way as private pensions. In both cases they increase with the increasing interest rate according to equation (3). The difference between these two systems is that the premium for the occupational pension depends mostly on the decision of the employer, the contributions and returns on investments are tax free, only benefits are taxed. On the other hand, the premium of private pension depends on the decision making of the beneficiary only and the contributions after tax income. As already presented in the paper of Di Giacinto and Vigna (2012) both, R_p and R_o, are mostly based on defined contribution principles. Since these risks are squarely on the workers' shoulders, they can only be mitigated by a proper portfolio structure of the pension pillars. Practically, the only available investment strategy is to include in the portfolio the asset whose cash flow is negatively correlated with the increasing interest rate.

Table 1: Comparison of yearly disbursement amounts according to the private pension scheme and loan model ERS for a 65-year old man, longevity is insured on the basis of mortality tables *DAV1994R*, value of fund and home value are both *100,000* EUR. The drawing period is 16 years

i	$a(65)$	$1.005\cdot a(65)$	$R_p = \frac{100,000}{1.005\cdot a(65)}$	YPA	$R_h(i)$	$R = R_p - R_h(i=5\%)$
2%	18.11	18.2	8,793	**4,560**	2,467	**3,933**
3%	16.31	16.39	9,762	**3,960**	3,215	**4,902**
4%	14.8	14.88	10,756	**3,480**	4,021	**5,896**
5%	13.53	13.59	11,771	**3,000**	4,860	**6,911**
6%	12.44	12.5	12,803	**2,520**	5,714	**7,943**
12%	8.56	8.6	19,128	**780**	10,715	**14,268**

From (9) the yearly amount of the disbursement from the ERS to beneficiary, denoted by YPA, can be calculated. In Table 1 we compare yearly disbursement from annuities purchased with 100,000 EUR at retirement and yearly disbursement from ERS which is based on residential property worth *VRE = 100,000* EUR at retirement. The drawing period is 16 years. Note that in the loan model the beneficiary also keeps his home, but in the funded pension it is assumed that he is also paying the yearly home lease (column 7 presents how in this case the income decreases if the discount factor of the home lease is 1.05 and the amortization period 80 years).

Theorem 1: If the interest rate decreases, the disbursement to the beneficiary from the loan model of ERS always increases.

Proof 1: From (9) it follows:

$$i\cdot MLA/\left[(1+i)^n-1\right]=YPA+prs(x,i,n)\cdot(YPA+MLA\cdot i)$$

$$YPA=MLA\cdot i\left\{1/\left[(1+i)^n-1\right]-prs(x,i,n)\right\}/(1+prs(x,i,n)) \quad (10)$$

The yearly amount of the disbursement from the ERS to beneficiary, denoted by YPA decreases with increasing positive i if

$$d\left\{i\left\{1/\left[(1+i)^n-1\right]-prs(x,i_a)\right\}\right\} \ /\ di<0 \Rightarrow$$
$$\Rightarrow 1/\left[(1+i)^n-1\right]-prs(x,i_a)-i\cdot n(1+i)^{n-1}/\left[(1+i)^n-1\right]^2<0 \tag{11}$$

From (11) it follows that the sufficient condition is:

$$(1+i)^{n-1}\left[1-(n-1)i\right]<1\,. \tag{11a}$$

Therefore, in the loan model of ERS, where it is always assumed $i>0$ and $n>1$, YPA always decreases with increasing positive i and, vice versa, YPA increases when i decreases.

Q.E.D.

The curves (3) and (10) are close to the linear curve (Fig. 3). While in pension and ERS schemes at (3) correlation coefficient $\rho_{R,i}$ is always close to 1, at (10) correlation coefficient $\rho_{YPA,i}$ is always close to -1. Therefore, we can expect that there exists an optimal portfolio which can substantially reduce the variance of portfolio (Markowitz, 1952).

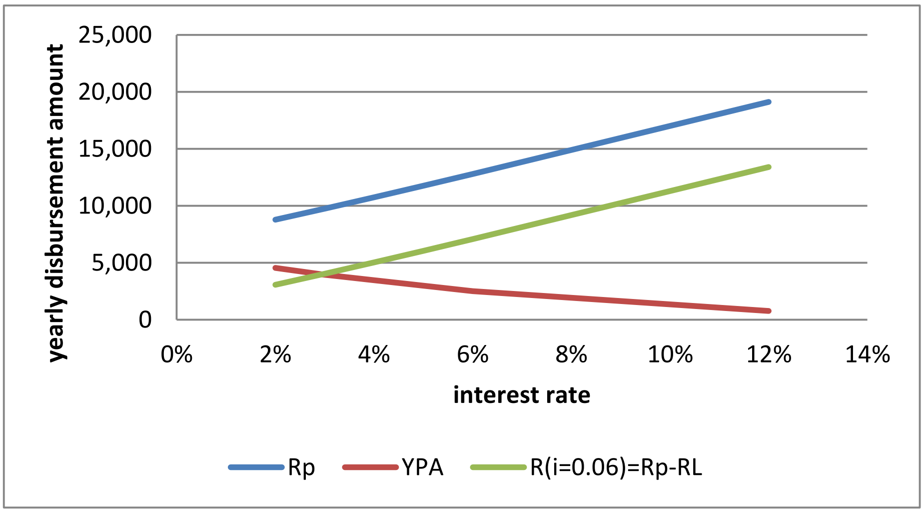

Figure 3: Private pension annuity: R_p, private pension reduced by the yearly amount of home lease: R_L, and yearlydisbursement amount from ERS denoted byYPA in dependence of interest rate

Let us denote the retirement income from the total portfolio of all 4 pillars by:

$$\tilde{R} = A+R_o+R_p+YPA=A+R+YPA \tag{12}$$

As the behaviour of R_o in dependence of interest rate is equal to the behaviour of R_p, we shall write $R = R_o + R_p$. Furthermore, let us denote by $\mu(\tilde{R})$ the expected value of $\tilde{R}$ and by $\sigma^2(\tilde{R})$ its variance influenced by the volatile interest rate. The individual worker, i.e. a pessimistic decision maker, cannot decide about the value of A and R_o, therefore these two incomes are just external parameters in his criterion function.

We shall denote the minimum income required for financing his lifestyle after retirement determined in advance by L. We suppose that the decision maker has the following two goals:

- that variance of total expected income due to volatile interest rate should be minimal,
- such a portfolio that his expected portfolio income $\mu(\tilde{R})$, reduced to a certain confidence limit (for example z $\sigma(\tilde{R})$), will be as close as possible to L.

Let us measure the importance of these two criteria by weight ψ $(0 \le \psi \le 1)$ for the first goal and by $1-\psi$ for the second goal. The main question here is: How much to invest in the private pension and how much to ERS from the income available during one's working period. Let us assume that when the pessimistic decision maker wishes to develop such a portfolio of pension wealth, his retirement income is as close as possible to his targeted lifestyle. These requirements are formalised as:

$$\begin{aligned}&\min_{\alpha}(\mu(\tilde{R}) - 3\sigma\ (\tilde{R}) - L\ )^2 = \min_{\alpha}(\mu(A + \alpha R(i) + (1-\alpha)YPA(i)) - \\ &-3\sigma(\alpha R(i) + (1-\alpha)YPA(i)) - L\ )^2\end{aligned} \tag{13}$$

and at the same time the variance is as low as possible when

$$\min_{\alpha} \sigma^2 \left[\alpha R(i) + (1-\alpha)YPA(i)\right] \tag{14}$$

The pessimistic decision maker, who does not wish to take the risk of the volatile interest rate and wishes to be able to assure an appropriate lifestyle, will put a higher or lower weight to each of these two criteria, which we shall formalize as:

$$\begin{aligned}&f_1^* = \min_{\alpha} \sigma^2 \left[\alpha R(i) + (1-\alpha)YPA(i)\right] \Rightarrow \alpha_1^* \\ &f_2^* = \min_{\alpha}\left[\mu(A + \alpha R(i) + (1-\alpha)YPA(i)) - 3\sigma(\alpha R(i) + (1-\alpha)YPA(i) - L\ \right]^2 \Rightarrow \alpha_2^* \\ &\alpha^* = \psi\alpha_1^* + (1-\psi)\alpha_2^* \qquad \psi \in [0,1]\end{aligned} \tag{15}$$

3.3 Minimization of the income variance by including ERS in the portfolio

Theorem 2: The minimum variance of the sum of income from all four pillars $\sigma^2(\tilde{R})$ as a criterion for the optimal portfolio of the pessimist who will work for 40 years (as it is the standard in many countries) and will retire at the standard retirement age (65)[3] is to be achieved if the share of the housing asset in the pension schemes is equal to:

$$\alpha = \left[\sigma^2(YPA) - \rho_{R,YPA}\sigma(R)\sigma(YPA)\right] / \left[\sigma^2(R) + \sigma^2(YPA) - 2\rho_{R,YPA}\sigma(R)\sigma(YPA)\right] \approx$$
$$\approx \left[\sigma^2(YPA) + 0.98\ \sigma(R)\sigma(YPA)\right] / \left[\sigma^2(R) + \sigma^2(YPA) + 1.96\sigma(R)\sigma(YPA)\right] \approx 0.28 \quad (16)$$

Proof 2: Because A does not depend on the interest rate and also from the fact that R and YPA are correlated it follows that also αR and $(1-\alpha)YPA$ are correlated, having the same correlation coefficient. Therefore the variance of this sum is:

$$\sigma^2(\alpha R + (1-\alpha)YPA) = \alpha^2\sigma^2(R) + (1-\alpha)^2\sigma^2(YPA) + 2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA) \quad (17)$$

On the bases of (3) and (9), i.e. regarding the distribution of the interest rate according to Schiller's time series (Figs. 1 and 2), we have simulated R and YPA and derived linear regression between the yearly values of R and YPA, which can be calculated from the data in Table 2. We got the following linear approximations:

$$YPA = 4{,}976k - 374\ i \qquad \rho_{YPA,i} > 0.9998$$
$$R_p = R_o = 6{,}622k + 1{,}036\ i \qquad \rho_{R_p,i} > 0.9998 \quad (18)$$

The correlation between YPA and R, which is also based on the asset of value expressed in $k \cdot 10^5$ EUR, is

$$YPA = 7352k - 0.36\ R, \quad (19)$$

and the correlation coefficient is equal to

$$\rho_{R,YPA} = 0.98 \quad (19a)$$

[3] http://ec.europa.eu/economy_finance/publications/european_economy/2012/pdf/ee-2012-2_en.pdf, Accessed December 2013.

Table 2: Distribution of 22-year moving averages for the yearly interest rate (1871 – 2012) and yearly income if all $k \cdot 10^5$ EUR is invested either in a funded pension or all in ERS when he retires

	i [%]	Relative frequency	Funded pension $R \ / k$	Lone model of ERS YPA / k
	3	0.16	6,097.5	2,475
	4	0.39	6,720	2,175
	5	0.15	7,357.5	1,875
	6	0.06	8,002.5	1,575
	7	0.05	8,655	1,350
	8	0.04	9,315	1,125
	9	0.15	9,975	937.5
μ	**5.2**	**mean**	7,485	1,875
σ^2	**28**	**variance**	1,683,506.25	266,256
σ	**5.3**	**standard deviation**	1,297.5	516

Therefore

$$2\alpha\sigma^2(R) - 2(1-\alpha)\sigma^2(YPA) + 2(1-2\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA) = 0$$

$$\alpha\left[2\sigma^2(R) + 2\sigma^2(YPA) - 4\rho_{R,YPA}\sigma(R)\sigma(YPA)\right] = 2\sigma^2(YPA) - 2\rho_{R,YPA}\sigma(R)\sigma(YPA) \quad (20)$$

while the second derivative is always positive. We can see that:

$$\sigma^2(R) + \sigma^2(YPA) - 2\rho_{R,YPA}\sigma(R)\sigma(YPA) > 0 \quad (21)$$

is valid for any α, if $i > 0$.

Finally, from (16) it follows that optimal α for minimal variance is:

$$\alpha = \left[\sigma^2(YPA) - \rho_{R,YPA}\sigma(R)\sigma(YPA)\right] / \left[\sigma^2(R) + \sigma^2(YPA) - 2\rho_{R,YPA}\sigma(R)\sigma(YPA)\right] \quad (22)$$

Replacing the values from Table 2 and from (19a) in (22), we can write:

$$\alpha = \left[\sigma^2(YPA) + 0.93\sigma(R)\sigma(YPA)\right] / \left[\sigma^2(R) + \sigma^2(YPA) + 1.86\sigma(R)\sigma(YPA)\right] =$$
$$k^2\left[266,256 + 0.98\cdot 1297.5\cdot 516\right] / \left\{k^2\left[1683506.25\cdot + 266256 + 1.96\cdot 1297.5\cdot 516\right]\right\} \quad (23)$$

From where it follows:

$$\alpha_1^* = 0.28$$

Q.E.D.

Remark 1: The optimal structure at the minimum of total income variance does not depend on k.

The calculation shows that the minimal volatility of income would be achieved when approximately 28% of asset is planned in private or occupational pension scheme, and 72% in ERS based on reverse mortgage. In this case the expected yearly income from occupational and private sources (occupational pension + private pension + ERS) would be:

$$\tilde{R} - A = 0.28\cdot k\cdot 7,485 + 0.72\cdot k\cdot 1,875 = 3,432.84\cdot k \quad (24)$$

Numerical example 1: Let us assume that a worker calculates that his expected accumulated asset in retirement from an occupational pension fund will be 30,000 EUR at the time of retirement, and his asset in private pension reserves and house would be 170,000 EUR altogether. It means that the expected total asset from the second, third and fourth pillars would be 200,000 EUR. In this case $k = 2$. Because the optimal share of funded pensions is 28%, he will add 26,000 EUR from his asset to the private pension schemes in order to achieve the minimal variance of his portfolio and he will invest 144,000 EUR in housing. From such a structure the sum of the expected yearly income from the second, third and fourth pillars would be 6865.68 EUR. The standard deviation in this case would be less than 1% and he would live in his house of a value of 0.72*200,000 EUR = 144,000 EUR until the end of his life, for which the expected implied yearly rent, according to the last column of Table 1, would be 4860 EUR*1.44 = 6998.4 EUR. Let us assume that his retirement income consists also of A = 12,000 EUR yearly from the PAYG system. In this case his total pension income would be 18,865.68 EUR; together with the implied rent his expected benefit is equal to 25,864 EUR per year and the standard deviation due to the volatile interest rate is less than 70 EUR.

Now we shall go back to the generalization of the problem and turn our attention to the question what to do if the portfolio in the structure calculated above does not fit to his expected lifestyle. How to build such a portfolio that his expected portfolio income $\mu(\tilde{R})$, reduced by $z\sigma(\tilde{R})$ where z is also his decision variable, will be as close as possible to L ?

3.4 Increasing the probability to cover the lifestyle of the pessimist

Theorem 3: The sufficient condition for the optimal structure of funded pension asset α and value of housing asset $(1-\alpha)$ in the portfolio of the pessimistic decision maker, who is planning his retirement lifestyle and assets to support it, when first employed (intends to work for a standard working period 40 years and retire at the standard retirement age of 65), is the intersection of the following two conditions:

$$\alpha = \frac{-\xi \pm \sqrt{\xi^2 - 4\omega\zeta}}{2\omega} \quad \wedge \quad \alpha \in [0,1]$$

where

$$\begin{aligned} &\omega = \left(z^2 3,262,001.85 - 31,472,100\right)k^2 \\ &\xi = \left[2z^2\right]\left[-922,375.8\right]k^2 + 11,220k\left[\left(L - A\right) - 1,875k\right] \\ &\zeta = z^2\sigma^2(YPA) - \left[\left(L - A\right) - \mu(YPA)\right]^2 \end{aligned} \tag{25}$$

Proof: The second condition in (15):

$$f_2^* = \min_{\alpha}\left[\mu(A + \alpha R(i) + (1-\alpha)YPA(i)) - 3\sigma(\alpha R(i) + (1-\alpha)YPA(i) - L\right]^2$$

is fulfilled if

$$\frac{d}{d\alpha}\left[\mu(A + \alpha R + (1-\alpha)YPA) - z\sqrt{\alpha^2\sigma^2(R) + (1-\alpha)^2\sigma^2(YPA) + 2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA)} - L\right]^2 = 0 \tag{26}$$

Let us write:

$$\sigma^2(\alpha R+(1-\alpha)YPA)=\alpha^2\sigma^2(R)+(1-\alpha)^2\sigma^2(YPA)+2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA)=$$
$$=\alpha^2(\sigma^2(R)+\sigma^2(YPA)-2\rho_{R,YPA}\sigma(R)\sigma(YPA))-$$
$$-2\alpha(\sigma^2(YPA)-\rho_{R,YPA}\sigma(R)\sigma(YPA))+\sigma^2(YPA)>0 \qquad (27)$$

Therefore from (26) we can write:

$$2\cdot\left[\mu(A+\alpha R+(1-\alpha)YPA)-z\sqrt{\alpha^2\sigma^2(R)+(1-\alpha)^2\sigma^2(YPA)+2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA)}-L\right]\cdot$$
$$\cdot\left\{\mu(R)-\mu(YPA)-\frac{z\left[\alpha(\sigma^2(R)+\sigma^2(YPA)-2\rho_{R,YPA}\sigma(R)\sigma(YPA))-(\sigma^2(YPA)+\rho_{R,YPA}\sigma(R)\sigma(YPA))\right]}{\sqrt{\alpha^2\sigma^2(R)+(1-\alpha)^2\sigma^2(YPA)+2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA)}}\right\}=0 \qquad (28)$$

From (27) and (28) the sufficient conditions (25) follow:

$$A+\alpha\mu(R)+(1-\alpha)\mu(YPA)-L=z\sqrt{\alpha^2\sigma^2(R)+(1-\alpha)^2\sigma^2(YPA)+2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA)}$$
$$z^2\left[\alpha^2\sigma^2(R)+(1-\alpha)^2\sigma^2(YPA)+2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA)\right]=\left\{\alpha\left[\mu(R)-\mu(YPA)\right]+\mu(YPA)-(L-A)\right\}^2$$

$$z^2\left[\alpha^2\sigma^2(R)+(1-\alpha)^2\sigma^2(YPA)+2\alpha(1-\alpha)\rho_{R,YPA}\sigma(R)\sigma(YPA)\right]=$$
$$=\left\{\alpha^2\left[\mu(R)-\mu(YPA)\right]^2+2\alpha\left[\mu(R)-\mu(YPA)\right]\left[\mu(YPA)-(L-A)\right]+\left[\mu(YPA)-(L-A)\right]^2\right\}$$

$$\alpha^2\left\{z^2\left[\sigma^2(R)+\sigma^2(YPA)-2\rho_{R,YPA}\sigma(R)\sigma(YPA)\right]-\left[\mu(R)-\mu(YPA)\right]^2\right\}+$$
$$+\alpha\left\{\left[2z^2\right]\left[-\sigma^2(YPA)+\rho_{R,YPA}\sigma(R)\sigma(YPA)\right]+2\left[\mu(R)-\mu(YPA)\right]\left[(L-A)-\mu(YPA)\right]\right\}+$$
$$+\left\{z^2\sigma^2(YPA)-\left[(L-A)-\mu(YPA)\right]^2\right\}=0$$
$$\omega\alpha^2+\xi\alpha+\omega=0$$
$$\alpha=\frac{-\xi\pm\sqrt{\xi^2-4\omega\zeta}}{2\omega}$$

Q.E.D.

Remark 2: The optimal structure of the pension pillars which cover the targeted lifestyle depends on k, the amount of disbursement needed from the second and third pillars $L-A$ and individual risk inclination described by *z*.

The set of possible solutions is only the set of L, A, k, z for which $\alpha\in[0,1]$.

Numerical example 2: Let us assume that a worker is calculating that his expected mathematical reserves from the occupational pension scheme will be 60,000 EUR at the time of retirement, and his asset in private pension reserves and house would be 240,000 EUR altogether. It means that the expected total asset from the second, third and fourth pillars would be 300,000 EUR. In this case $k=3$. For this pessimistic decision maker, who is planning a standard working period of 40 years and a standard retirement age at 65, we can write (based on Table 2) the expected values and variance for the yearly income if all the money is put either in funded schemes or all in housing:

Table 3: Expected yearly annuities if all money is invested either in a funded pension or all in housing $k=3$

	$R=R_o+R_p$	YPA
$\mu(.)$	22,455	5,625
$\sigma^2(.)$	10,101,037.5	1,597,536
$\sigma(.)$	3,892.5	1,548

From Table 3 the following can be calculated:

$$\omega = z^2\left\{\sigma^2(R)+\sigma^2(YPA)-2\rho_{R,YPA}\sigma(R)\sigma(YPA)\right\}-\left[\mu(R)-\mu(YPA)\right]^2=\left(z^2 3,262,001.85-31,472,100\right)k^2$$

$$\xi=\left[2z^2\right]\left[-\sigma^2(YPA)+\rho_{R,YPA}\sigma(R)\sigma(YPA)\right]+2\left[\mu(R)-\mu(YPA)\right]\left[(L-A)-\mu(YPA)\right]=$$

$$=\left[2z^2\right]\left[-922,375.8\right]k^2+11,220k\left[(L-A)-1,875k\right]$$

$$\zeta=z^2\sigma^2(YPA)-\left[(L-A)-\mu(YPA)\right]^2=266,256k^2z^2+\left[(L-A)-1,875k\right]^2$$

If a young pessimistic worker expects his retirement income from the PAYG system in the amount of A = 12,000 EUR per year, and he is expecting $\mu(R_o)$ = 4,491 EUR of occupational pension per year, this is not enough to finance his targeted lifestyle, as his desired minimal standard (here named targeted standard) of life after the retirement requires 21,000 EUR (alternative: 17,070 EUR) per year. Therefore, $\mu(\alpha(R_o+R_p)+(1-\alpha)YPA)$ should be equal or higher than 9,000 EUR (alternative: 5,070 EUR). According to the forecasting, in occupational pension fund, private pension scheme and housing assets he can accumulate altogether 300,000 *EUR (k=3)* when retired, therefore we have to calculate the following values, presented in Table 4, while $\alpha = \frac{-\xi + \sqrt{\xi^2 - 4\omega\zeta}}{2\omega}$ is out of range $[0,1]$. It is also presented how the optimal structure is changed when the asset at retirement increases for 55,000 EUR.

Table 4: Calculation of the optimal $\alpha = \alpha_2^*$, $L - A = 9{,}000$;

k	z	ω	ξ	ζ	$\alpha = \frac{-\xi - \sqrt{\xi^2 - 4\omega\zeta}}{2\omega}$
3	0.5	-275909396	109451809	11989701	0.486
	1	-253890883	96999736	13786929	0.492
	2	-165816833	47191442	20975841	0.525
	3	-19026750	-35822380	32957361	0.677
	3.2	17377190.5	-56409807	35928778	0.870
3.55	0.5	-386349796	87541786	6332036.87	0.284
	1	-355517762	70105424	8848655.3	0.285
	2	-232189627	359978.09	18915129	0.286
	3	-26642735.4	-1.16E+08	35692585.2	0.289
	3.2	24332893.7	-1.45E+08	39853394.4	0.290

Table 4: Calculation of the optimal $\alpha = \alpha_2^*$, $L - A = 5{,}070$

2	0.5	-122626398.2	27776048.4	2008656	0.284
	1	-112840392.6	22241793.6	2807424	0.285
	2	-73696370.4	104774.4	6002496	0.286
	3	-8456333.4	-36790257.6	11327616	0.289
	3.2	7723195.776	-45940225.5	12648245.76	0.290

Remark 3: The more reliable his targeted life style, the higher the percentage of asset in private pension schemes.

The calculation of (23) shows that the minimal volatility of income would be achieved when approximately 28% of the asset is allocated in private or occupational pension schemes and 72% in ERS based on reverse mortgage. But probability to achieve the targeted lifestyle in this structure is decreasing. Therefore a pessimistic decision maker will be weighting between the lower volatility and higher reliable of income for financing his lifestyle. Depending on his preferences and loss aversion, he will choose:

$$\alpha^* = \psi\alpha_1^* + (1-\psi)\alpha_2^*(z); \quad \psi \in [0,1] \quad .$$

Let us choose the level of his pessimism at *z = 2*, where *z* depends on probability accepted to fall under lifestyle line. In this case it follows from Table 4 the value of $\alpha_2^*(2) = 0.525$. The pessimistic decision maker will choose $\alpha^* = 0.28\psi + 0.525(1-\psi)$ Therefore :

$$\alpha^* = -0.245\psi + 0.525; \quad \psi \in [0,1] \quad .$$

From here it follows that the decision maker who is the most loss-averse will choose $\psi = 0$ (will make a plan to invest up to 47.5% in housing in his work period, i.e. up to 142,500 EUR, while he will invest in private pension 97,500 (=300,000-60,000-142,500) EUR.

From Table 4 there follows also the remarks 4 and 5:

Remark 4: With the lower value of z (a less pessimistic decision maker), the optimal structure becomes more stable.

Remark 5: If $(L - A)/k = 5{,}070/k$ remains constant and close to 2,500 EUR, the optimal share of ERS income in the pension portfolio of the pessimist remains constant and very close to the optimal structure at the minimum variance of the total portfolio.

4 Conclusions and plan for further research

As mentioned also in some other CEJOR and EJOR papers, the crisis of national PAYG public pension systems all over Europe, caused by the ageing of population, is forcing governments of most European countries to cut pension benefits of future generations drastically, while encouraging developments of other three pillars of the pension systems, making future retirees more pessimistic. In a few decades, future retirees will have to rely heavily on the provision of those pillars if they want to maintain the pre-retirement standard of living, that is, if the social policy in Europe will not change. As stated by Di Giacinto and Vigna in one of the CEJOR papers, pension funds can be either defined benefit (DB) or defined contribution (DC); however, DC schemes place all risks of the turbulent economic environment on the shoulders of the workers who contribute to the schemes and who are the final beneficiaries. Therefore, in this article Equity Release Schemes have been studied in the context of a pessimistic decision maker.

ERS transform fixed assets in owner occupied dwellings into liquid assets for private pensions. We have shown that the interest rate variation, which can reduce the income of older persons even below the targeted lifestyle line, has a significantly smaller impact on welfare of the elderly if funded pensions, which depend on a volatile interest rate and which have a positive covariance with the interest rate, are combined with the ERS loan model, where the correlation coefficient is negative. Because of volatility of the interest rate the proper combination of pensions and dynamics of ERS drawings has to be planned to decrease the volatility of combined cash flows deriving from different pension pillars. But the pessimistic decision maker also tries to ensure, with high probability, that his target lifestyle will be fully financed with retirement income from all 4 pillars. Therefore, he is making a trade-off between the low volatility of income and the high probability that his lifestyle will not be affected.

Due to the negative correlation of cash flows from funded pensions with disbursements from ERS, the volatility induced by the volatile interest rate can be reduced even to 2.5% of volatility of income from the funded pension schemes, when only 28% of the asset dependent on the interest rate is invested in funded pension schemes and the rest in ERS (at the standard working period and the standard retirement age). But at such a structure, the probability of falling below the line of one’s lifestyle after retirement could be higher, especially when the targeted lifestyle is higher, while the investments in the pension pillars throughout the working period are the same. For example, to assure a high probability that the lifestyle will not be affected, it is advisable to raise the percentage of assets in funded pension schemes up to 52,5% (z=2, $(L - A)/k = 3,000$) or even higher if this ratio is lower – but then the volatility of the annuities increases.

Let us sum up the five important remarks for an optimal structure of the pension portfolio creation:

1. The optimal portfolio structure of pension wealth using the minimum variance criterion does not depend on the amount of wealth accumulated in this portfolio.
2. The optimal structure at the targeted lifestyle depends on the level of total investments in the pension portfolio, the amount of annuities needed from the second and third pillars, and personal risk inclination.
3. The more reliable the targeted lifestyle, the higher the share of assets in funded private pension schemes.
4. The more pessimistic a decision maker, the less stable the optimal structure.

5. If the amount of yearly annuities from the funded pillars and ERS divided by k is close to 2,500 EUR $\left[(L-A)/k \approx 2{,}500\right]$, the optimal structure remains constant and very close to the optimal structure at the minimum variance of the total portfolio and does not depend much on the loss aversion of the decision maker.

The article considers the investment decision in the pension portfolio of a pessimistic decision maker who first enters employment, who is planning to work for a standard working period of 40 years and retire at the standard retirement age (65), as also set out in the Slovenian Pension and Disability Insurance Act (ZPIZ-2)[4]. Hence, the parameters in the model have been derived under these standard assumptions, but in further research we could include a changed working period and a changed retirement age, which could be interesting due to the ageing European population and, as a result, the changing legislation on the pension provision in EU. Indeed, the changed parameters in pension schemes would have implications for the optimal structure of the pension portfolio for the pessimistic decision maker also. In this research, we have investigated the optimal portfolio strategy for DC pension plans and ERS in the pension portfolio of four pillars, minimizing the decline from the amount of annuities needed to cover the targeted lifestyle expenses. Here, the interest rates are assumed to be stochastic; however, these schemes were not considered to be exposed to inflation. The annuities received from the different pillars in the annuitization phase were exposed to volatility of the interest rate, but were not assumed to be inflation-indexed to protect the annuitant's purchasing power after retirement.

PAYG systems and ERS do not depend on inflation; however, in addition to the risk of the interest rate, the investment period is too long for DC pension plans to neglect the effect of inflation. In the long run, the accumulated inflation could severely damage the real payoff of the pension plan (Han and Hung, 2012). These two authors consider the stochastic program where the inflation-indexed bonds are introduced as an instrument that delivers a defined disbursement indexed by the inflation at its maturity. To consider this instrument in our model is our main duty in the next months.

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